Heavenly Mathematics: Observing the Sun and the Moon from Different Parts of the World

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1 Introduction

I was born in Oslo, Norway, did my Ph.D. at UC Berkeley in California, and have worked at the National University of Singapore for 16 years. For the last few years, I have taught a class called "Heavenly Mathematics & Cultural Astronomy" ([3]). The first part of the course covers basic positional astronomy, focusing on what the Sun, the Moon and the stars look like from different parts of the world. Many astronomy books take a "high-latitude centric" point of view, but I try to be "hemispherically correct" and consider the points of views of observers in the Northern Hemisphere, the Southern Hemisphere and the Tropics. The rest of the course is dedicated to applications of these ideas with a cultural flavor, like calendars ([2]) and navigation. In this talk I will give some samples from the early part of the course.

Modern humans have become alienated from many aspects of nature. Before the time of electricity, everybody was keenly aware of the motion of the Sun and the Moon and a lot of what I cover in my course was common knowledge 100 years ago. However, many concepts require subtle geometrical understanding. I hope that after this talk, you will appreciate new aspects of nature and see the centrality of mathematics in understanding the world around us!

2 Where does the Sun rise?

When I tell my students that the Heel Stone at Stonehenge marks the rising position of the Sun at the summer solstice, some of them are very impressed. They seem to think that this is evidence that Stone Age people possessed computers or were helped by aliens. They don't realize that all you have to do to find out where the Sun rises at the summer solstice is to get up early in the morning on that day!

We all know that the Sun rises in the east, right? Well, what do we mean by east? Do we mean exactly at the east point on the horizon, or do we mean somewhere in the eastern part of the horizon? The rising position of the Sun changes in the course of the year, and the way it changes depends on your latitude. In fact, one of the homework in my Heavenly Mathematics class is to take three pictures of the Sun rising or setting in the course of two months to see how the rising or setting position of the Sun changes.

In order to explain this, we must start by recalling some basic facts from astronomy. For excellent introductions to spherical astronomy, see the books by Evans ([13]), Kaler ([15]) and Rogers ([17]).

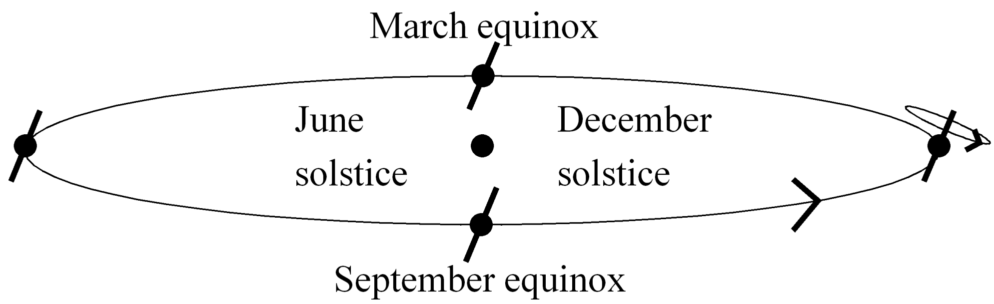

Figure 1: The ecliptic plane

The Earth revolves counterclockwise (when viewed from above the North Pole) around the Sun in an elliptical orbit (Figure 1). However, the eccentricity of the orbit is so small that we can often assume for simplicity that the orbit is a circle.

The plane of the orbit is called the *ecliptic* plane. The word ecliptic is derived from the fact that eclipses can only occur when the Moon crosses this plane. The Earth rotates counterclockwise (when viewed from above the North Pole) around an axis that is tilted approximately 23.5° to the normal to the ecliptic plane. This is called the *obliquity*.

Notice how astronomers make a distinction between *revolving* and *rotating*. An object rotates around an axis that passes through it, but it revolves around an outside object.

People often say that Ptolemy believed that the Sun moved around the Earth, while Copernicus and Kepler showed that the Earth moves around the Sun. That is not a good way of expressing the difference. Whether the Earth moves around the Sun or the Sun moves around the Earth is primarily a question of your point of view. If you claim that the Sun doesn't move around the Earth, you need to go out more! If you want to understand why the heavenly bodies move the way they do, you need to understand Newton's theory of gravity, which explains why the Earth moves around the Sun. But for an observer on Earth who wants to describe the motion of the Sun and the Moon, it makes sense to take a geocentric point of view.

The real difference between the geocentric and heliocentric points of view is the planetary theory. While there is nothing wrong in saying that the Sun moves around the Earth, it is wrong to say that the planets move around the Earth. The Copernican revolution consisted in making the planets move around the Sun.

Early astronomers realized that the motion of the Sun along the ecliptic was not uniform. It follows from Kepler's Second Law that the Earth moves faster along the orbit when it is close to the Sun. This makes sense, since gravity is the driving force. The point in the Earth's orbit that is closest to Sun is called *perihelion* and the point that is furthest from the Sun is called *aphelion*. The Earth is at perihelion around January 4 and at aphelion around July 4.

When the Earth's axis tilts towards the Sun, there is summer in the North Temperate Zone. The points in the orbit where the projection of the Earth's axis onto the ecliptic plane point directly towards the Sun (Figure 2) used to be called the summer and winter *solstices*. But since I teach in Singapore and we are close to Australia, I prefer the 'hemispherically correct" terms June and December solstices. They occur around June 21 (or 20) and December 21 (or 22) in the Greenwich time zone. They move back and forth because of leap years. At the March and September *equinoxes* around March 20 (or 21) and September 22 (or 23), also called the vernal (spring) and autumnal (fall) equinoxes, the radial line from the Sun to the Earth is perpendicular to the Earth's axis. These four points are called the *seasonal markers*.

The above definitions are of course not the way people in ancient civilizations determined the solstices and equinoxes. A simple way was to look at how the rising position of the Sun changes over the course of the year. The Sun rises due east at the equinoxes, at which time day and night are equally long. The word "equinox" is derived from a Latin word meaning equal night. Strictly speaking, the day is a bit longer at the time of the equinox. This is because sunrise is the time when the top of the Sun reaches the horizon, while sunset is the time when the top of the Sun goes below the horizon. In addition, refraction bends the image of the Sun upwards by about one degree near the horizon. After the March equinox, the rising position of Sun moves north until the Sun reaches its

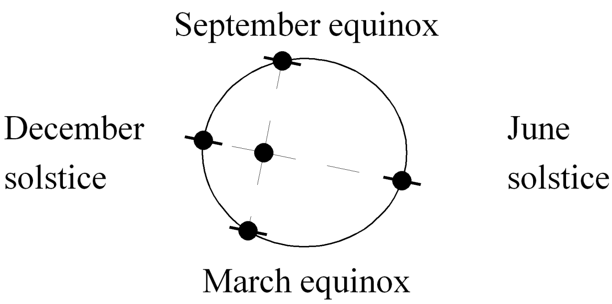

Figure 2: Solstices and equinoxes

northernmost rising position at the June solstice. Here the Sun seems to stand still, and the word “solstice” is derived from the Latin word “solstitium”, which means standing Sun. The Sun then starts moving south again, passing due east at the time of the September equinox, and reaching its southernmost position at the December solstice. Figure 3 shows the daily path of the Sun at the time of the solstices and equinoxes for observers in Beijing, Singapore and Sydney.

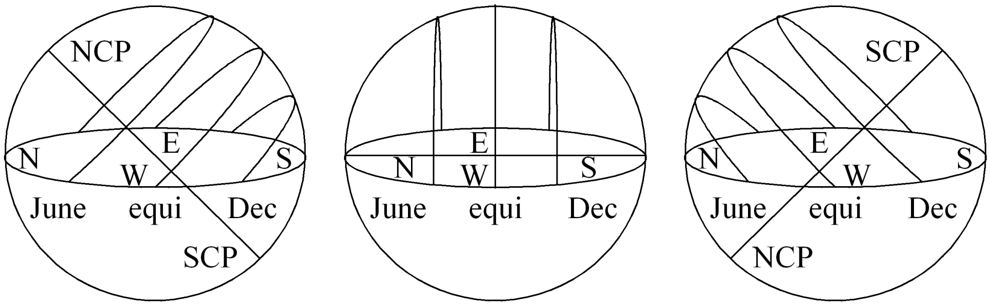

Figure 3: The daily path of the Sun at the time of the solstices and equinoxes for observers in Beijing (40°N), Singapore (1°N) and Sydney (34°S)

Notice how the path of the Sun on different days form circles around the axis between the north celestial pole (NCP) and the south celestial pole (SCP). Strictly speaking, the daily path of the Sun is not a circle. The declination (angular distance from the celestial equator) is changing all the time, and this prevents the daily path from being a closed curve. The path of the Sun in the course of the year is sometimes referred to as a “spiral of circles” ([17]).

Notice also that at the Equator, the difference between the northernmost and the southernmost rising positions of the Sun is $2 \times 23.5 = 47°$, while as we move towards the Arctic Circle, the difference increases to 180°. To understand this, look at Figure 3 and see how the position of the three circles change as your latitude changes. At the Arctic Circle, the northernmost circle will be completely above the horizon and the southernmost circle will disappear below the horizon.

Just south of the Arctic Circle, the rising and setting positions of the Sun in late June will be just a little bit east and west of the north point on the horizon, while the rising and setting positions of the Sun in late December will be just a little bit east and west of the south point on the horizon.

We will need some terminology regarding different regions on the Earth. The Arctic is the region north of the Arctic Circle at 66.5°N and the Antarctic is the region south of the Antarctic Circle at 66.5°S. These are the regions on the Earth where there will be midnight Sun part of the year, total darkness part of the year and a period with daily sunrise and sunset in between. From Figure 3, you can see that at the Equator, the Sun will be directly overhead at the time of the equinoxes. At the time of the June solstice, the Sun will be directly overhead the (small) circle of latitude corresponding to 23.5°N. This is called the Tropic of Cancer. Similarly, the circle of latitude corresponding to 23.5°S is called the Tropic of Capricorn. The area between the two tropics is called the Tropics. (It will be clear from the context whether tropics (or Tropics) refers to the region or is the plural of the two circles.) The Tropics is therefore the region of the Earth where, at some time of the year, the Sun will be directly overhead. The area between 23.5°N and 66.5°N is called the North Temperate Zone and the area between 23.5°S and 66.5°S is called the South Temperate Zone.

3 Which way does the Sun move in the course of the day?

The Earth rotates counterclockwise (when viewed from above the North Pole) around the Sun. So which way will the Sun move in the course of the day? Well, that depends on where you are. If you are in the Arctic or the North Temperate Zone, the Sun will move clockwise, while if you are in the Antarctic or the South Temperate Zone, the Sun will move counterclockwise. If this is not clear, try standing on your head, or at least bend your head down!

But what about the Tropics? As you can see from Figure 3, the Sun will cross the horizon at almost a right angle. This has a number of interesting consequences. In particular, there will be very little dawn or dusk. It goes from daylight to darkness in as little as 24 minutes. Why 24 minutes? There are several definitions of twilight, but civil twilight is when the Sun is between zero and six degrees below the horizon. Since it takes 24 hours for the Sun to cover 360°, it moves 15° each hour and it takes four minutes to move one degree If the Sun crosses the horizon vertically, then the Sun's altitude will change by six degrees in 24 minutes.

This also raises another question: Why do clocks go clockwise? Before me-

chanical clocks were invented, people used sundials, and in the North Temperate Zone the shadow of a vertical stick will move clockwise in the course of the day. So when people invented mechanical clocks, it was only natural to make the hands of the clock move the same direction as the shadow of the sundial.

But what if sundials had been invented in the Southern Hemisphere? Would all clocks have gone counterclockwise? Well, that is only partially true. They would have gone the other direction, but we would still have called it clockwise!

Astronomers prefer to talk about eastward or westward motion in the sky, instead of clockwise or counterclockwise motion. In some cases this clearly is a better terminology. Observers in Beijing, Singapore and Sydney will all agree that the Sun moves westward in the course of the day. However, the Beijing observer will say that it moves clockwise, the Sydney observer will say counterclockwise, and to the Singapore observer it makes more sense to say that the Sun goes straight up and down.

Personally, I used to have a hard time understanding whether westward meant east to west by way of north or east to west by way of south. I finally realized that it meant east to west by way of the visible part of the sky. However, I still don't understand how knowing that the Sun moves westward is going to help you if you're at the North Pole and the Sun is in the east!

4 What time does the Sun rise?

It is well know that in the Northern Hemisphere, the Sun will rise early in the summer and late in the winter. In the Southern Hemisphere it is the other way around. But what about the Equator? Most people would guess that at the Equator the Sun would rise at 6 a.m. all year round. That is not correct! First of all we have to worry about the time zone and the longitude. Singapore is at longitude 104°E, and since each time zone corresponds to $360°/24 = 15°$ and $7 \times 15° = 105°$, you would expect Singapore to use the UTC + 7 time zone. (UTC means Coordinated Universal Time, which is the modern name for GMT, Greenwich Mean Time). However, for various political reasons Singapore follows UTC + 8 time ([9]). In other words, Singapore is on permanent daylight saving time. So does the Sun rise at 7 a.m. then? No, because it is only in the middle of the time zone that the zone time corresponds to local mean time. Since one hour corresponds to 15°, it follows that each degree corresponds to 4 minutes. Singapore is at 104°E with UTC + 8, so it uses a time that corresponds to a longitude 16° further east. This means that the Sun rises $16 \times 4\text{m} = 1\text{h}4\text{m}$ later than the clock indicates. We would therefore guess that the Sun would rise around 7:04 a.m. every day. Unfortunately, our guess is still wrong, but now the explanation is more subtle. The observant reader may have noticed that I used the term local "mean" time

earlier. We need to understand the difference between “mean” and “true” time, the equation of time and the analemma. I will not go into the details, but just give a rough outline.

Suppose you took a picture of the Sun every week at a fixed time. What kind of curve would the Sun trace out in the course of the year? If you look at Figure 3, you would probably, after some thinking, decide that the Sun would move up and down along a straight line parallel to the Earth’s axis. If you took the picture at noon, you would expect a curve up and down along the meridian. The meridian is the great semicircle between the north and south points on the horizon that passes through the zenith, the point straight above you. We would expect the Sun at noon to be on the meridian between altitude $c - 23.5°$ and $c + 23.5°$. (Here we replace $c + 23.5°$ by $180° - (c + 23.5°)$ if $c + 23.5° > 90°$.) If we do it in the morning or afternoon, we would expect it to be a line segment spanning 47° along a line parallel to the Earth’s axis.

However, as the famous analemma picture by Dennis di Cicco ([12]) from Sky & Telescope in Figure 4 show, we instead get a strange figure eight curve called the *analemma*.

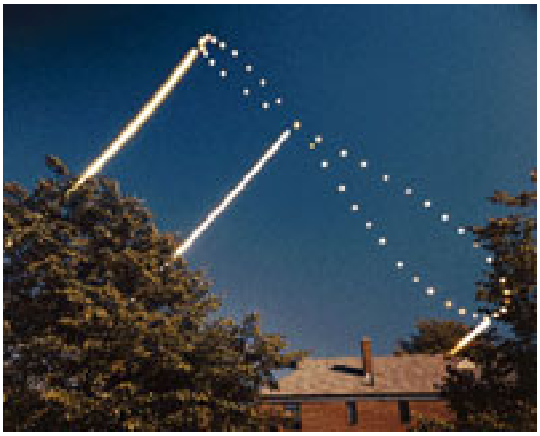

Figure 4: Analemma in Boston at 8:30 a.m. by Dennis di Cicco ([12])

The picture traces the position of the Sun at 8:30 a.m. in the course of a year. The 44 solar images were exposed at 8:30 a.m. every seven days or the next clear day using a digital alarm clock. Human judgment regarding the weather was necessary, and the camera was armed manually on the morning of the exposure. However, more than half of the exposures were made with no one present.

The three streaks were created by leaving the camera shutter open from sunrise until 5 minutes before the regular exposure. Fortunately the streaks were not interrupted by clouds.

The streak to the crossover point of the figure eight could have been made in

the middle of April or at the end of August. The April date was out of the question, since there would be no leaves on the tree then. Without leaves to block the Sun, its strongly exposed image would appear to be between the camera and the tree. Note that this did happen for the winter streak.

To complete the picture, a foreground exposure was made in September with the Sun in the southwest (note the chimney's shadow on the roof) to create a dark sky in which the sun images would stand out.

The angle that the axis of the analemma makes with the horizon is about 42°, which is the latitude of di Cicci's house in Watertown outside Boston.

This picture was taken in 1978–1979. Di Cicci had made an earlier attempt in 1977–1978, but the camera had not been aimed properly, and the top part of the analemma had been cut off!

What would this look like if we had picked another time of the day? Figure 5 shows Analemmas over Athens at 10:00:00 UTC + 2, 12:28:16 UTC + 2 and 15:00:00 UTC + 2, photographed by Anthony Ayiomamitis ([10]).

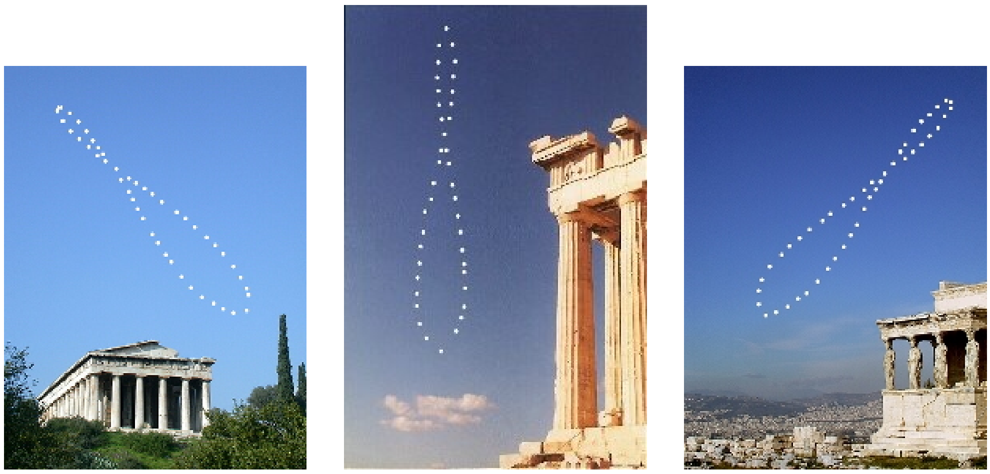

Figure 5: Analemmas over Athens at 10:00:00 UTC + 2, 12:28:16 UTC + 2 and 15:00:00 UTC + 2, photographed by Anthony Ayiomamitis

Why did he take the noon picture at 12:28:16 UTC + 2? The longitude of Athens is about 23°E and it lies in the UTC + 2 time zone. However, since one degree corresponds to four minutes, the Sun will cross the meridian in Athens about $(30-23)*4 = 28$ minutes after it crosses the meridian at the center of the time zone at longitude 30°E.

As you can tell from di Cicco's description of how he took the picture, it is hard to photograph the analemma, but there is another place you easily see the analemma. On many globes you will see an analemma in the Pacific Ocean as in Figure 6.

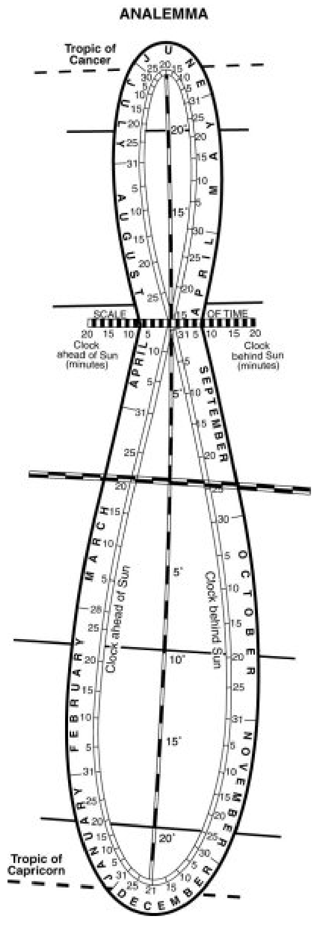

Figure 6: Analemma on a globe

The analemma is often pictured on globes because you can also see the analemma on the ground. Suppose you place a vertical stick in the ground, and mark the shadow of the stick on the ground at noon every day. What kind of curve will the shadow trace out in the course of the year? Well, we must be careful with our definition of noon. *True time* is time according to the sundial, while *mean time* is time according to our clock. In the same way, *true noon* is when the Sun crosses the meridian, while *mean noon* is 12:00:00 p.m.

If you look at the shadow at true noon, the shadow will always point along the north-south line. If you live north of the Tropic of Cancer, the shadow will always point north. If you live south of the Tropic of Capricorn, the shadow will always point south. If you live in the Tropics, the shadow will point south part of the year, on two days of the year there will be no shadow at noon, and part of the year the shadow will point south. In any case, the true noon shadow curve will always be a straight line.

However, the mean noon shadow curve will be like the curve in Figure 6, i.e., the analemma. But notice that the ground analemma and the sky analemma are reversed. As you can see from the marking on the ground analemma in Figure 6, November is in the eastern part of the curve. If we added markings on the sky analemma in Figure 5, November would be in the western part of the curve. This is just a consequence of how the shadow is formed, and you can simply think of November as always being on the right side, assuming that north is up.

But I still haven't explained why we get the figure eight shape instead of the straight line we would have expected. For full details you may refer to [8], but I will give a brief outline here.

Mean time is our usual clock time, while *true time* is time measured by the Sun, as in sundial. *True noon* is the (mean) time when the Sun crosses the meridian, while *mean noon* is simply 12:00 (in mean time). The *true Sun* is the usual Sun, while the *mean Sun* is an imaginary object moving along the celestial equator at the same speed as the true Sun moves along the ecliptic. Please note that true noon is not the true time when the true Sun crosses the meridian. So if the Sun is 15 min fast, the true noon is at 11:45, but the true time of mean noon is 12:15.

True noon and mean noon are moments in time, while true time and mean time are "clocks". The mean time is your usual clock, the true time is a variable clock with days of different length.

We will also need the concept of *local mean time*. This is the same as mean time, but adjusted for your longitude. If you live in the middle of a time zone and there is no daylight saving time, it is simply the time that your clock shows.

The equation of time is defined by

$$\text{equation of time} = \text{true time} - \text{local mean time}.$$

The word equation here means an adjustment, namely the adjustment we must

make to the mean time to get the true time, i.e.,

$$\text{true time} = \text{local mean time} + \text{equation of time.}$$

In old books, and also in modern French books, the opposite sign is used, so

$$\text{local mean time} = \text{true time} + \text{equation of time.}$$

This made sense in the past, since people then relied on sundials, and mean time had to be computed from true time. Once mechanical clocks became more reliable than sundials, it made sense to change to the modern convention. The analemma is a graphical representation of the equation of time.

Let us now try to understand the reasons for the difference between true time and mean time. The daily motion of the Sun across the sky caused by the Earth's rotation is westward. However, the motion of the Sun along the ecliptic caused by the Earth's revolution is eastward. The motion caused by the revolution is only $1/365$ times as fast as the motion caused by the rotation, so the net effect is that the Sun moves westward together with the stars and the rest of the sky, but since $24\text{h}/365 \approx 24\text{h}/360 = 4\text{m}$ it lags about 4 minutes behind the stars each day.

There are two reasons why true time is different from mean time. Since the orbit of the Earth around the Sun is an ellipse, the revolution of the Earth around the Sun is faster near the time of perihelion around January 4. We observe this as the Sun revolving around us more quickly along the ecliptic. Since the motion along the ecliptic caused by the revolution slows down the motion caused by the rotation, the net effect is that the true solar day, i.e., the time from one true noon to the next is longer. At the time of perihelion, the eccentricity makes the true solar day about 8s longer than the mean solar day. However, the cumulative effect adds up to about 8m.

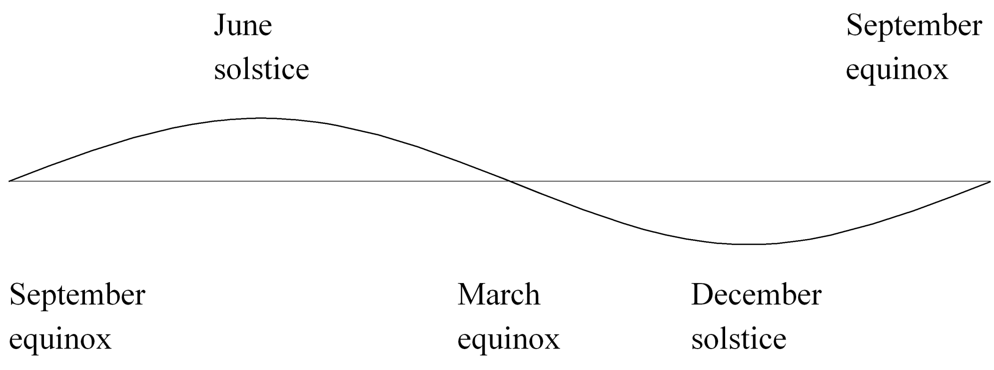

Figure 7: The declination of the Sun in the course of the year

The other contributing factor is the obliquity of the Earth's axis. If we ignore the effect we just considered, and assume that the speed of the Earth along the

ecliptic is constant, then the speed of the Sun along the graph in Figure 7 is constant. But the clock is related to the rotation of the Earth, so we can imagine a "mean Sun" that moves to the left along the x-axis, which represents the celestial equator, while the "true Sun" moves along the graph, which represents the ecliptic. If the true Sun moves at constant speed, then its projection onto the x-axis will move at variable speed. At the time of the equinoxes the true Sun and the mean Sun will be together, and at the time of the solstices, the true Sun and the mean Sun will be directly above each other. However, right after the equinoxes, the true Sun will fall behind, because part of its motion is sideways. At the time of the solstices, the true will have an advantage. Why? This is not clear from Figure 7, but you have to remember that Figure 7 is flat version of the motion of the true Sun of the celestial sphere. If you look at a globe, the meridians are closer together at higher latitude, so the same daily distance along the ecliptic corresponds to more degrees along the celestial equator. The increased speed of the true Sun at the solstices and decreased speed at the equinoxes, makes the solar day longer around the solstices and shorter around the equinoxes. Why? Remember that the daily motion is westward, so when the Sun yearly eastward motion is faster, the net effect is to make the true solar day longer. At the time of the solstices, the obliquity makes the true solar day about 20s longer than the mean solar day. However, the cumulative effect adds up to about 10m.

Let us now see how the two effects combine. Notice that the obliquity creates two fast periods near the solstices, and two slow periods near the equinoxes. The eccentricity creates a fast period near perihelion, which is near the December solstice, and a slow period near aphelion, which is near the June solstice. The combined effect is that the maximum value of the equation of time is about 16m30s around November 3 and the minimum value is about -14m15s around February 11. The obliquity effect causes the figure eight shape of the analemma, and eccentricity effect makes the figure eight asymmetrical.

Let us now look at some graphical applications of this. The concept of "analemma-rise" is due to Keith and Stage ([16]). Let us start with an observer at the Equator, for example Singapore at 1°N. Figure 8 shows the image of the analemma at 6 a.m. at the Equator. You can see that the earliest Sunrise will be on around November 3 and the latest sunrise will be around February 11. (The dates are only approximate, and may move back and forth by a day because of leap years.)

Suppose we now move north to latitude 5°N, for example Penang in Malaysia. This will tilt the analemma, and we see from Figure 9 that we now will get two dates on which the Sun rises earliest, May 23 and October 22.

We now move north to latitude 12°N, for example near Bangkok at 13°N. We then see from Figure 10 that there is a 50 days period between August 22 and October 10 when the variation in the sunrise in within an interval of 20 seconds!

If we leave the Tropics and travel north to Beijing at around 40°N, the earliest

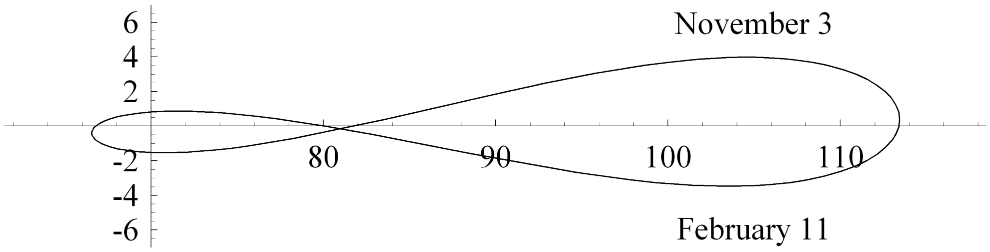

Figure 8: 6 a.m at the Equator

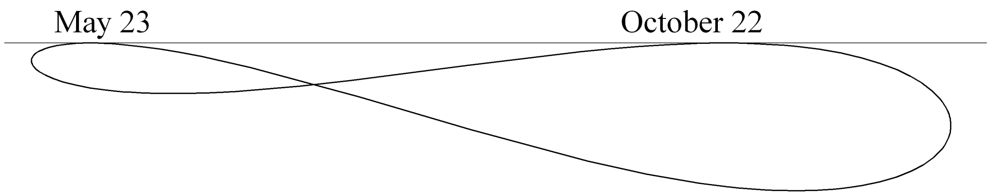

Figure 9: Earliest sunrise at 5°N

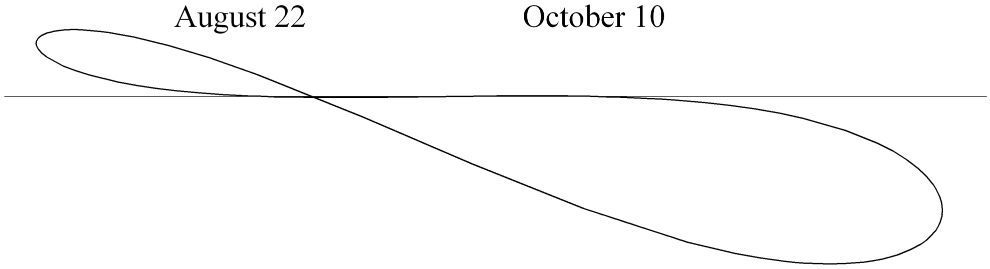

Figure 10: Stable sunrise at 12°N

sunrise will be on June 14 and the latest on January 5. These are dates that will make more sense to people who live in the North Temperate Zone. They associate early sunrises with summer. But most people who live in the North Temperate Zone would have guessed that the earliest sunrise would occur on the longest day. Instead we see that it occurs more than a week before the June solstice at around June 21.

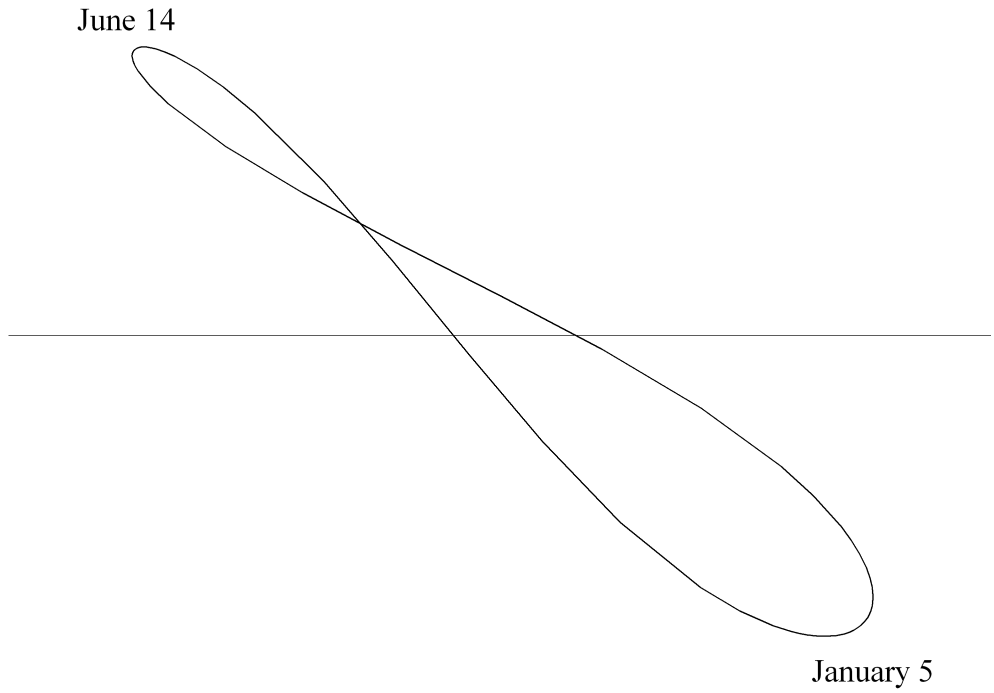

Figure 11: Earliest and latest sunrise at 40°N

Finally we travel north to Oslo at around 60°N. Now the earliest sunrise will be on June 19 and the latest on December 27. At such a high latitude, the earliest sunrise and the longest day almost coincides.

These are just some highlights of a fascinating topic. One way of summarizing it is to say that in the Tropics the sunrise and sunset times are primarily determined by the equation of time and the analemma, while a higher latitude the tilt of the analemma caused by the latitude is the decisive factor. For more information, please see [18, 19].

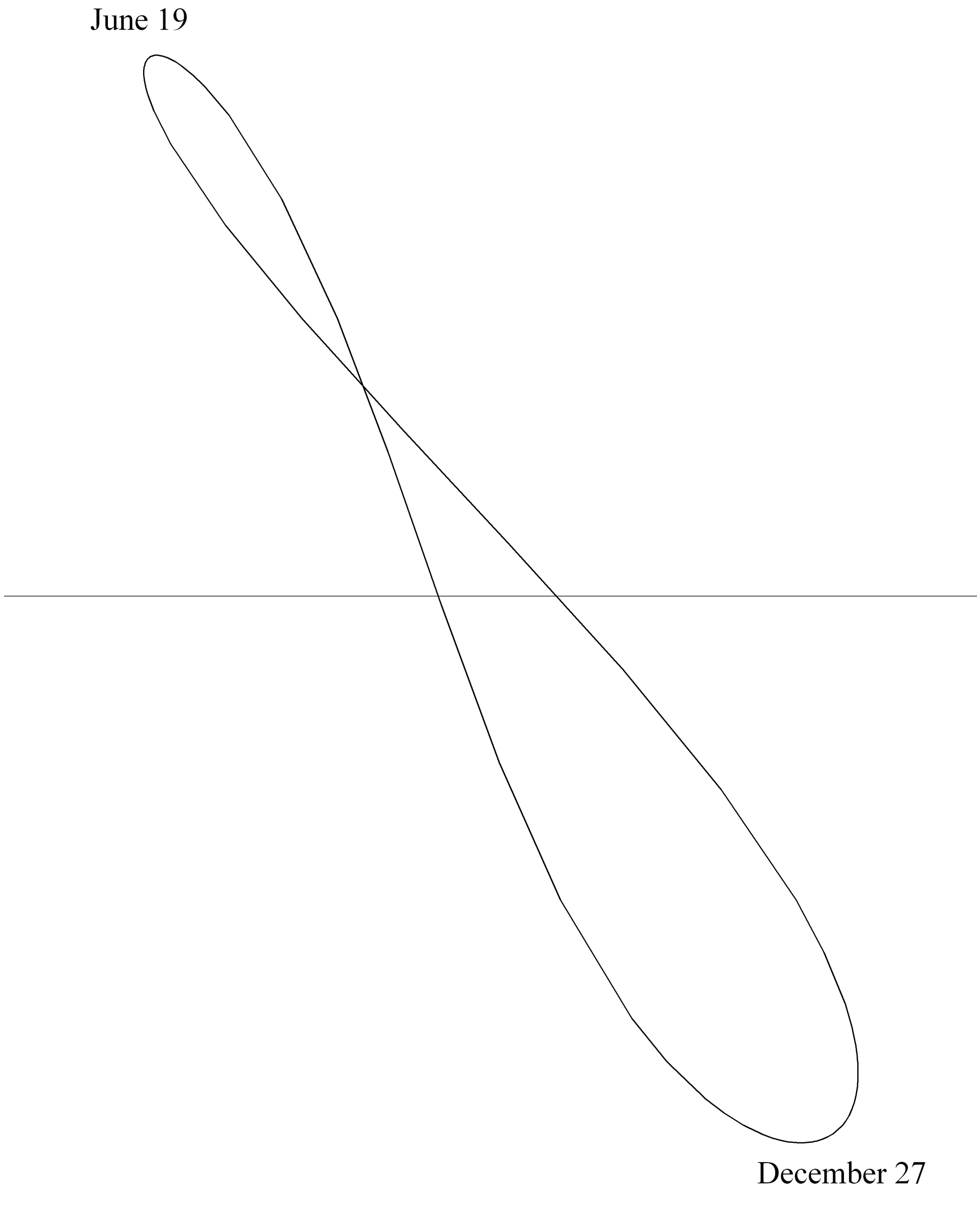

Figure 12: Earliest and latest sunrise at 60°N

5 What does the orbit of the Moon look like?

What does the orbit of the Moon around the Sun look like? Most people think it will look like the picture on the left, but in fact it will look like a 13-gon with smooth edges, like the picture on the right. There are no loops or changes in the sign of the curvature.

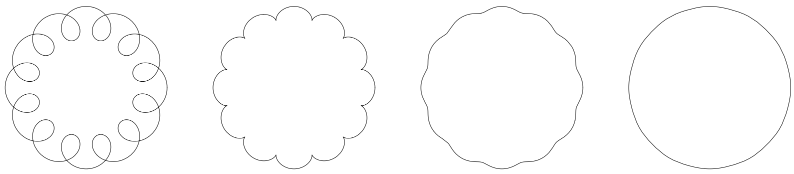

Figure 13: What does the orbit of the Moon around the Sun look like?

There are several ways to see this. Since the eccentricities are small, we can assume that the orbits of the Earth around the Sun and the Moon around the Earth are both circles. The radius of the Earth's orbit is about 400 times the radius of the Moon's orbit. The Moon makes about 13 revolutions in the course of a year. The speed of the Earth around the Sun is about 30 times the speed of the Moon around the Earth. That means that the speed of the Moon around the Sun will vary between about 103% and 97% of the speed of the Earth around the Sun. In particular, the speed of the Moon around the Sun will never be negative, so the Moon will never loop backwards.

I like to visualize this in the following way. Imagine you're driving on a circular race track. You overtake a car on the right, and immediately slow down and go into the left lane. When the other car passes you, you speed up and overtake on the right again. You will then be making circles around the other car, but when seen from above, both of you are driving forward all the time and your path will be convex.

Another approach is to compute the gravitational forces involved. It can be shown that the Sun's pull on the Moon is about twice the Earth's pull on the Moon. It follows that the Moon's orbit is primarily determined by the gravitation pull from the Sun, so the orbit of the Earth will always curve towards the Sun. For more info, please see [6, 11, 14].

6 What does a waxing crescent look like?

If you see a crescent Moon near the horizon, how can you tell if it is a waxing or a waning Moon? In some languages there are mnemonics to help you remember

this, but it is important to realize that they all depend on your latitude.

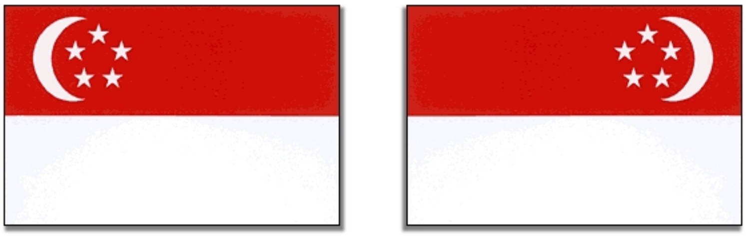

Figure 14: The front and back of the Singapore flag

The Singapore flag in Figure 14 was adopted in 1959. The crescent Moon represents a young nation on the ascendant, and also the close relationship at the time between Singapore with its ethnic Chinese majority and Malaya with its Muslim majority. On the front we see a left crescent, while on the back we see a right crescent. So what will a waxing crescent look like in Singapore, a left or a right crescent? The answer is neither! Near the Equator, a waxing crescent will not look like the crescent on the flag. It will look like the crescent on the coat of arms in Figure 15!

Figure 15: The Singapore Coat of Arms

In the Northern Hemisphere, a waxing crescent will be a right crescent while an waning crescent will be left crescent. In the Southern Hemisphere a waxing crescent will be a left crescent while a waning crescent will be a right crescent. This is illustrated in Figure 16. In the Northern Hemisphere the Moon moves clockwise across the sky, while in the Southern Hemisphere it moves counter-clockwise across the sky. So what happens near the Equator? The Moon will move in a straight line when seen from above, or straight up and down along an east to west line when seen from the ground. So both the waning and the waxing crescent will be a bottom crescent! In order to tell them apart, you have to think about whether the Moon is in the east or the west and what time it is. Notice that

the crescent faces the Sun, so there can never be a top crescent. Notice that this only applies to crescents near the horizon. If the crescent is higher in the sky, there is no natural way to define left or right.

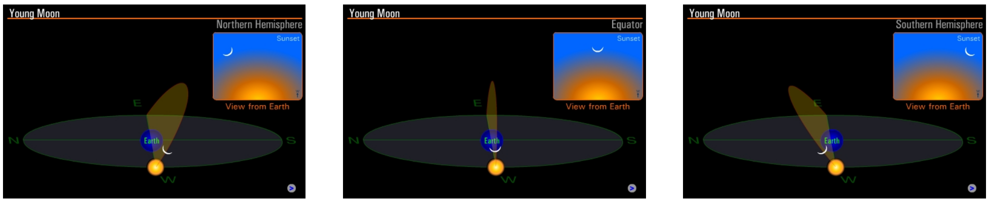

Figure 16: What does a waxing crescent look like in different parts of the world?

If you look at a young Moon in Singapore, you will see that it is not always straight down. There are two reasons for this. The Moon is on the ecliptic, and when it crosses the horizon, the angle between it and the daily path of the Sun, which is a small circle parallel to the celestial equator, can be up to 23.5°. The angle will be maximal at the time of the equinoxes, while at the time of the solstices, the ecliptic and the celestial equator will be parallel in the sense that their tangent lines will be parallel.

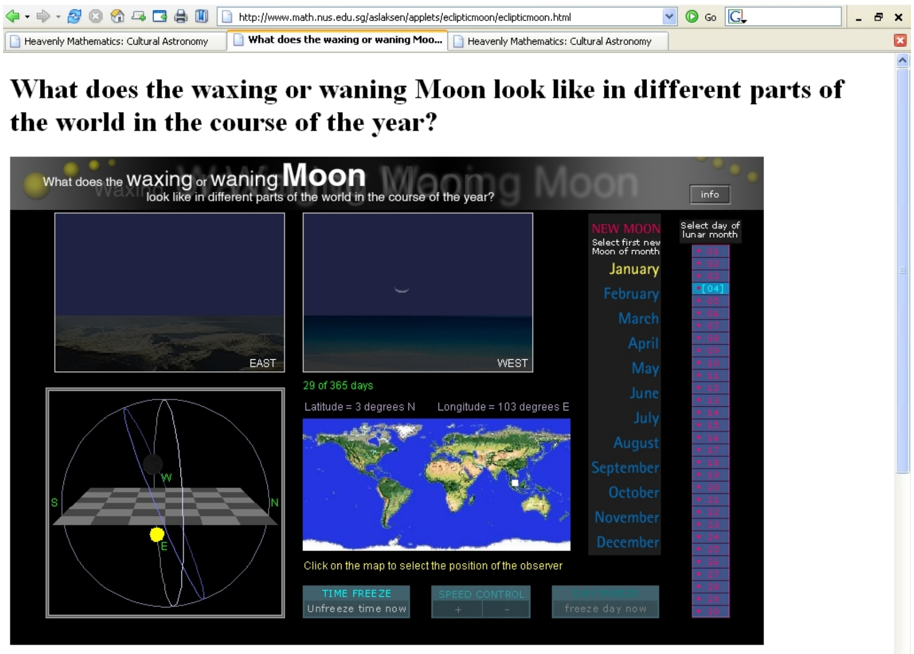

Figure 17: My lunar crescent applet

This is illustrated in an applet on the web page for my course on Heavenly Mathematics ([1]). I have included a screen shot in Figure 17. In addition the

latitude of the Moon can be up to five degrees. (Notice that on the celestial sphere, latitude measures angular separation from the ecliptic, and not from the celestial equator.) For more details, please see ([4, 7])

To conclude let me point out another astronomical problem with the Singapore flag. You can never see stars inside the Moon!

I hope you have enjoyed this lecture and that wherever in the world you go, you will pay attention to the Sun and the Moon and use your mathematical skills to understand their fascinating behavior!

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